Effects of compressibility on a laminar wall jet

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SUMMARY

The effects of compressibility on a radial laminar wall jet are investigated. On the assumption that the coefficient of viscosity is proportional to the temperature, it is shown that a similarity solution for the velocity distribution exists, which is expressible directly in terms of the corresponding solution for an incompressible wall jet. For arbitrary Prandtl number the energy equation is studied in detail and solutions are obtained for a variety of temperature conditions.

1. INTRODUCTION

The term 'wall jet' was introduced by Glauert (1956) to describe the flow due to a jet of air spreading out over a plane surface, either radially or in two dimensions. With compressibility neglected, Glauert studied the velocity distribution in a wall jet for both laminar and fully turbulent flow, and for laminar flow found an exact solution of the boundary-layer equations in the form of a similarity solution. This solution should be asymptotically approached whatever the initial form of the jet near the axis.

The present paper considers the influence of compressibility on a laminar radial wall jet. The effects of viscous dissipation, wall temperature conditions and of the initial temperature of the fluid in the wall jet are analysed. Some of the results of this study have been briefly referred to by Glauert (1957).

Glauert (1956) proved that, in an incompressible wall jet, the 'flux of exterior momentum flux' is constant; this forms the basic starting point of his theory. It is here shown how this result can be generalized to compressible flow, the flux of exterior momentum flux being constant when the viscosity is proportional to the temperature.

In the boundary-layer equations of momentum, continuity and energy, the stream function may be conveniently chosen as independent variable in place of the coordinate normal to the wall. This transformation was first used for incompressible flow by von Mises (1927) and later by von Kármán & Tsien (1938) for compressible flow. Under the simplifying assumption that the viscosity is proportional to the temperature, it is possible to solve the momentum and energy equations independently, the momentum equation becoming identical with the corresponding equation for incompressible flow, and the same solution being applicable. The interpretation of the results in terms of geometrical coordinates differs, and is fully investigated. In particular it is shown that the skin friction has the same value as in the corresponding incompressible flow.

Still retaining the simple viscosity–temperature law, but imposing no restrictions on the Prandtl number, solutions of the energy equation are obtained. These solutions describe the effects of viscous heating, wall temperature and initial jet temperature on the velocity and temperature profiles, both when there is no heat transfer across the wall and when the wall is maintained at a constant temperature. If the assumption that the Prandtl number is unity is made, the well-known Crocco relation between the temperature and the velocity provides a particular solution of the energy equation, being a special form of the solutions obtained in the more general analysis. When a more accurate viscosity–temperature law is assumed it is possible to develop solutions of the momentum and energy equations in series form, though the details are not set out here.

In practice, wall jets will almost always be turbulent. No attempt is made in this paper to extend Glauert's analysis for turbulent flow, since it is not easy to predict how the eddy viscosity and diffusivity will vary with temperature. However, the general nature of the changes in the velocity profiles in the laminar case may perhaps serve as an indication of the sort of modifications to be expected for turbulent compressible flow.

Glauert (1956) shows that there is a close analogy between wall jets in two and three dimensions, the same velocity profile occurring in each case. Similarly all the results obtained here, with simple changes, become applicable to compressible plane wall jets.

2. Equations of motion

On the boundary-layer approximation, the momentum, continuity and energy equations governing a compressible, laminar, radial wall jet flowing over a plane wall are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{2.1}$$

$$\frac{\partial}{\partial x}(\rho x u) + \frac{\partial}{\partial y}(\rho x v) = 0, \tag{2.2}$$

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = \frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu \frac{\partial T}{\partial y}\right) + \frac{\mu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{2.3}$$

The boundary conditions are $u = v = 0$ at $y = 0$; $u \to 0$, $T \to T_\infty$ as $y \to \infty$, together with a condition on T at $y = 0$, $x = 0$.

Here x and y denote distances along and normal to the wall, x being measured from the jet axis, u and v the corresponding velocity components, ρ the density, T the temperature and μ the coefficient of viscosity. The specific heat at constant pressure C_p and the Prandtl number σ are assumed constant. The subscript ∞ is used to denote values at $y = \infty$. On the boundary-layer approximation the pressure is uniform everywhere and hence the equation of state implies that

$$\rho T = \text{const.} \tag{2.4}$$

For an incompressible wall jet, Glauert established an integral relation which he interpreted as saying that the flux of exterior momentum flux is constant. We can follow the same procedure as Glauert to obtain a corresponding result for compressible flow. Multiply equation (2.1) by x and integrate with respect to y between the limits y and ∞, using the condition that $u \to 0$ as $y \to \infty$; then, since

$$\begin{aligned}\int_y^\infty \rho x v \frac{\partial u}{\partial y}\,dy &= [\rho x v u]_y^\infty - \int_y^\infty u \frac{\partial}{\partial y}(\rho x v)\,dy \\ &= -\rho x v u + \int_y^\infty u \frac{\partial}{\partial x}(\rho x u)\,dy\end{aligned}$$

using (2.2), we have

$$\frac{\partial}{\partial x}\int_y^\infty (\rho x u^2)\,dy - \rho x v u + x\mu \frac{\partial u}{\partial y} = 0. \tag{2.5}$$

Multiplying (2.5) by $\rho x u$ and integrating with respect to y between the limits 0 and ∞, we have

$$\begin{aligned}\frac{\partial}{\partial x}\int_0^\infty \rho x u \left\{\int_y^\infty \rho x u^2\,dy\right\} dy - \int_0^\infty \frac{\partial}{\partial x}(\rho x u)\left\{\int_y^\infty \rho x u^2\,dy\right\} dy - \\ - \int_0^\infty \rho^2 x^2 v u^2\,dy + \int_0^\infty \rho\mu x^2 u \frac{\partial u}{\partial y}\,dy = 0. \end{aligned}\tag{2.6}$$

From the continuity equation, the second term of (2.6) is

$$\left[\rho x v \int_y^\infty \rho x u^2\,dy\right]_0^\infty + \int_0^\infty \rho^2 x^2 v u^2\,dy.$$

Now at $y = 0$, $u = v = 0$, therefore equation (2.6) reduces to

$$\frac{\partial}{\partial x}\int_0^\infty \rho x u \left\{\int_y^\infty \rho x u^2\,dy\right\} dy + \int_0^\infty \rho\mu x^2 u \frac{\partial u}{\partial y}\,dy = 0. \tag{2.7}$$

For incompressible flow, where ρ and μ are constants, the second term of (2.7) is zero. For compressible flow it will vanish only in certain circumstances. In particular it will do so if $\rho\mu$ is a function of the velocity u, or is constant. In these cases (2.7) reduces to

$$\int_0^\infty \rho x u \left\{\int_y^\infty \rho x u^2\,dy\right\} dy = F, \tag{2.8}$$

where the constant F is the flux of exterior momentum flux. It may be noted, that as defined here, the constant F differs by a factor ρ^2 from the definition used by Glauert (1956).

The continuity equation (2.2) implies the existence of a stream function ψ such that

$$\left.\begin{aligned}\rho x u &= \frac{\partial \psi}{\partial y}, \\ \rho x v &= -\frac{\partial \psi}{\partial x}.\end{aligned}\right\} \tag{2.9}$$

Following von Mises, we now take x and ψ as the independent variables, and hence write

$$\left.\begin{aligned}\left(\frac{\partial}{\partial x}\right)_y &= \left(\frac{\partial}{\partial x}\right)_\psi - \rho v x\left(\frac{\partial}{\partial \psi}\right)_x,\\ \left(\frac{\partial}{\partial y}\right)_x &= \rho u x\left(\frac{\partial}{\partial \psi}\right)_x.\end{aligned}\right\} \tag{2.10}$$

Substituting (2.10) into the momentum and energy equations (2.1) and (2.3) we have

$$\frac{\partial u}{\partial x} = x^2\frac{\partial}{\partial \psi}\left(\rho\mu u\frac{\partial u}{\partial \psi}\right), \tag{2.11}$$

$$\frac{\partial T}{\partial x} = \frac{x^2}{\sigma}\frac{\partial}{\partial \psi}\left(\rho\mu u\frac{\partial T}{\partial \psi}\right)+\frac{\rho\mu x^2 u}{C_p}\left(\frac{\partial u}{\partial \psi}\right)^2. \tag{2.12}$$

On introducing dimensionless variables by writing

$$\left.\begin{aligned}&u = U\bar{u}, \quad x = \nu_\infty \bar{x}/U, \quad \psi = \rho_\infty \nu_\infty^2 \bar{\psi}/U,\\ &\rho = \rho_\infty \bar{\rho}, \quad \mu = \mu_\infty \bar{\mu}, \quad T = T_\infty \bar{T},\end{aligned}\right\} \tag{2.13}$$

where U is an arbitrary constant velocity and ν the kinematic viscosity, we find that equations (2.11) and (2.12) become

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \bar{x}^2\frac{\partial}{\partial \bar{\psi}}\left(\bar{\rho}\bar{\mu}\bar{u}\frac{\partial \bar{u}}{\partial \bar{\psi}}\right), \tag{2.14}$$

$$\frac{\partial \bar{T}}{\partial \bar{x}} = \frac{\bar{x}^2}{\sigma}\frac{\partial}{\partial \bar{\psi}}\left(\bar{\rho}\bar{\mu}\bar{u}\frac{\partial \bar{T}}{\partial \bar{\psi}}\right)+\frac{U^2}{C_p T_\infty}\bar{\rho}\bar{\mu}\bar{x}^2\bar{u}\left(\frac{\partial \bar{u}}{\partial \bar{\psi}}\right)^2. \tag{2.15}$$

Since ρT is constant by (2.4) ,we have, if $\mu \propto T$, $\rho\mu = \text{const.} = \rho_\infty \mu_\infty$ or

$$\bar{\rho}\bar{\mu} = 1. \tag{2.16}$$

Using (2.16) we see that equations (2.14) and (2.15) become

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \bar{x}^2\frac{\partial}{\partial \bar{\psi}}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{\psi}}\right), \tag{2.17}$$

$$\frac{\partial \bar{T}}{\partial \bar{x}} = \frac{\bar{x}^2}{\sigma}\frac{\partial}{\partial \bar{\psi}}\left(\bar{u}\frac{\partial \bar{T}}{\partial \bar{\psi}}\right)+\frac{U^2}{C_p T_\infty}\bar{x}^2\bar{u}\left(\frac{\partial \bar{u}}{\partial \bar{\psi}}\right)^2. \tag{2.18}$$

3. Velocity distribution

The density $\bar{\rho}$, and hence the temperature $\bar{T}$, does not appear explicitly in the momentum equation (2.17), and so (2.17) may be solved independently of the energy equation (2.18). Moreover a particular solution of equation (2.17) must be the similarity solution, obtained by Glauert (1956), of the corresponding incompressible equation, that is

$$\bar{u} = \tfrac{3}{4}f'(\eta)\bar{x}^{-3/2}, \tag{3.1}$$

where

$$f(\eta) = \bar{\psi}\bar{x}^{-3/4} \tag{3.2}$$

and $f(\eta)$ satisfies the equation

$$\left.\begin{aligned}f''' + ff'' + 2f'^2 = 0,\\ f(0) = f'(0) = 0, \quad f(\infty) = 1.\end{aligned}\right\} \tag{3.3}$$

The boundary condition at the outer edge of the wall jet only requires $f'(\infty) = 0$, but writing $f(\infty) = 1$ involves no loss of generality, since the velocity U in (2.13) remains arbitrary. Glauert obtained the solution of equation (3.3) in the form

$$f = g^2, \qquad \eta = \log\frac{\sqrt{(1+g+g^2)}}{(1-g)} + \sqrt{3}\tan^{-1}\frac{\sqrt{3}g}{(2+g)}. \tag{3.4}$$

More recently, Glauert (1957) has shown that

$$\bar{u} = \tfrac{3}{4}f'(\eta)(\bar{x}^3+l^3)^{-1/2}, \tag{3.5}$$

where

$$f(\eta) = \bar{\psi}(\bar{x}^3+l^3)^{-1/4}, \tag{3.6}$$

is also a solution of the momentum equation (2.17), l being an arbitrary constant length. Since the velocity given by equation (3.5) remains finite as $\bar{x} \to 0$, this solution may have some significance even near the axis. We shall make use of this latter solution of the momentum equation.

Although the velocity function $f'(\eta)$ is the same in the incompressible and compressible problems, the interpretation of the variable η differs. Connection of η with the physical variable y can be made as follows. Using (2.9) and (2.13)

$$y = \int_0^\psi \frac{d\psi}{\rho x u} = \frac{\nu_\infty}{U}\int_0^{\bar{\psi}} \frac{\rho_\infty}{\rho \bar{u}\bar{x}}\, d\bar{\psi}. \tag{3.7}$$

Substituting $\rho_\infty/\rho = T$ and introducing $\bar{u}$ and η according to equations (3.5) and (3.6) we obtain, from equation (3.7)

$$y = \left(\frac{U(x^3+l^3)^3}{\nu_\infty x^4}\right)^{1/4} \frac{4}{3}\int_0^\eta T\, d\eta. \tag{3.8}$$

Thus, in order to obtain the velocity in terms of the physical variable y, we see that the temperature distribution throughout the wall jet must be known. The temperature distribution will be the subject of §4. The skin friction is independent of the temperature distribution when $\mu \propto T$, since it is given by

$$\tau_w = \left(\mu\frac{\partial u}{\partial y}\right)_{y=0} = \frac{9}{16}\mu_\infty f''(0)\left(\frac{\nu_\infty x^4}{U(x^3+l^3)^5}\right)^{1/4} \tag{3.9}$$

which has the same value as for incompressible flow.

The arbitrary velocity U may be expressed in terms of the flux of exterior momentum flux F defined by equation (2.8). Substituting the values of (3.5) and (3.8), we obtain

$$F = \frac{3}{40}\frac{\nu_\infty^4 \rho_\infty^2}{U}. \tag{3.10}$$

Proceeding exactly as in the corresponding incompressible case, if the wall jet is the result of a free jet impinging on a flat plate, a rough estimate of the magnitude of F may be obtained from the conditions in the free jet as

$$F = \tfrac{1}{2} \times (\text{typical velocity}) \times (\text{mass flow/radian})^2. \tag{3.11}$$

The constant length l can be estimated by equating the maximum velocity at $x = 0$, given by (3.5), to the maximum velocity in the impinging jet.

Alternatively, if the wall jet is formed by fluid flowing out radially, with uniform velocity U_0 and temperature T_0, from beneath a circular disc of radius a and at a distance h from the wall, then equation (2.8) may be evaluated at $x = a$ to give

$$F = \frac{U_0^3 h^2 a^2}{2T_0^2}. \tag{3.12}$$

In this case l can be estimated by requiring the maximum velocity at $x = a$, given by (3.5), to be U_0.

4. Temperature distribution

In order to determine the temperature distribution, the energy equation (2.3) must be solved for T. When the Prandtl number σ is unity, a particular solution of (2.3) is the Crocco relation

$$T + \frac{u^2}{2C_p} = A + Bu, \tag{4.1}$$

where A and B are constants. Since $u \to 0$ as $y \to \infty$, A must have the value T_∞. The value of B depends on the degree of heating in the initial jet. In particular the solution with $B = 0$ will be appropriate for a jet issuing from a reservoir of fluid maintained at the same temperature, T_∞, as the fluid outside the wall jet. In this case the heat transfer across the wall is seen to be zero. If we suppose the wall jet to be formed by fluid flowing out uniformly from beneath a circular disc as in § 3, then equation (4.1) can be evaluated at $x = a$, giving

$$B = \frac{T_0 + \frac{1}{2}(\gamma-1)M_0^2 - 1}{U_0} \cdot T_\infty, \tag{4.2}$$

where γ is the ratio of the specific heats and $M_0 = U_0/a_\infty$, a_∞ being the velocity of sound outside the wall jet.

We shall now seek solutions of the energy equation for arbitrary Prandtl number. The simple solutions, for Prandtl number unity, given by (4.1) occur as special cases of these more general solutions.

When the assumption $\mu \propto T$ is made, the energy equation takes the form given in equation (2.18), a linear equation for T. Since $\bar{u}(\bar{x}, \eta)$ is known, it is convenient to change the independent variables again from $(\bar{x}, \bar{\psi})$ to $(\bar{x}, \eta)$, where η is defined by equation (3.6). To do this we have the transformation formulae

$$\left.\begin{aligned} \left(\frac{\partial}{\partial \bar{x}}\right)_{\bar{\psi}} &= \left(\frac{\partial}{\partial \bar{x}}\right)_\eta - \frac{3}{4}\frac{f}{f'}\frac{\bar{x}^2}{(\bar{x}^3+l^3)}\left(\frac{\partial}{\partial \eta}\right)_{\bar{x}}, \\ \left(\frac{\partial}{\partial \bar{\psi}}\right)_{\bar{x}} &= \frac{1}{f'(\bar{x}^3+l^3)^{1/4}}\left(\frac{\partial}{\partial \eta}\right)_{\bar{x}}. \end{aligned}\right\} \tag{4.3}$$

Substituting for $\bar{u}$ from equation (3.5) the final form of the energy equation (2.3) is

$$\frac{\partial^2 T}{\partial \eta^2} - \sigma\left(\frac{4}{3}\frac{(\bar{x}^3+l^3)}{\bar{x}^2} f' \frac{\partial T}{\partial \bar{x}} - f\frac{\partial T}{\partial \eta}\right) = -\sigma \frac{9}{16}\frac{U^2}{C_p T_\infty}\frac{f''^2}{(\bar{x}^3+l^3)}. \tag{4.4}$$

This equation (4.4) is the form in which we shall study the energy equation. A boundary condition at the wall, $\eta = 0$, must be specified. We shall consider only the two cases of a perfectly conducting wall, over which the temperature is maintained at a constant value, and of a thermally insulated wall, at which the heat transfer is zero. The outer boundary condition is $\bar{T} = 1$. A condition at $\bar{x} = 0$ must also be specified. This is related to the temperature of the initial jet. When the solution for $\bar{T}$ has been found, the correspondence between η and the physical variable y may be obtained from equation (3.8).

A particular integral of equation (4.4) may be found in the form

$$\bar{T} = 1 + C_0(\bar{x}^3 + l^3)^{-1}\theta_0(\eta). \tag{4.5}$$

Equation (4.4) also has an infinity of complementary functions of the form

$$\bar{T}_n = C_n(\bar{x}^3 + l^3)^{-\alpha}\theta_n(\eta), \tag{4.6}$$

where θ_n satisfies the equation

$$\theta_n'' + \sigma(f\theta_n' + 4\alpha f'\theta_n) = 0. \tag{4.7}$$

By adding to the particular integral (4.5) suitable complementary functions of the type (4.6), the temperature conditions we wish to be able to prescribe can be satisfied.

The effects of viscous heating, wall temperature and initial heating will be discussed separately, and the solutions appropriate for both the required conditions at $\eta = 0$ will be determined.

Viscous heating

Let us first examine the temperature distribution due to viscous heating in the wall jet. As we have seen, we require a solution of equation (4.4) in the form (4.5), where, if we choose

$$C_0 = -\frac{9}{16}\frac{U^2}{C_p T_\infty}, \tag{4.8}$$

the equation to be satisfied by $\theta_0(\eta)$ is

$$\theta_0'' + \sigma(f\theta_0' + 4f'\theta_0) = \sigma f''^2, \tag{4.9}$$

where

$$\left.\begin{array}{lll} & \theta_0(\infty) = 0 & \\ \text{and either} & \theta_0(0) = 0 & \text{for constant wall temperature } T_\infty, \\ \text{or} & \theta_0'(0) = 0 & \text{for a thermally insulated wall.} \end{array}\right\} \tag{4.10}$$

When $\sigma = 1$, $\theta_0(\eta) = \frac{1}{2}f'^2(\eta)$ satisfies all the conditions (4.10), a result indicated by (4.1). For arbitrary σ, no solution of equation (4.9) has been found in closed terms. The equation has been integrated numerically for $\sigma = 0{\cdot}72$, the value appropriate for air.

When the wall jet is flowing over a thermally insulated wall, the wall temperature is, from equations (4.5) and (4.8)

$$\bar{T}_w = 1 - \frac{9}{16}\frac{U^2}{C_p T_\infty}\frac{\theta_0(0)}{(\bar{x}^3 + l^3)}. \tag{4.11}$$

The numerical integration gave $\theta_0(0) = 0{\cdot}0060$.

When the wall is maintained at a constant temperature T_∞, the rate at which heat is transferred across the wall, per unit area, is

$$Q_w = -\left(\lambda\frac{\partial T}{\partial y}\right)_{y=0} \tag{4.12}$$

$$= \frac{27}{64}\lambda_\infty\left(\frac{\nu_\infty^{13}x^4}{U^5(x^3+l^3)^7C_p^4}\right)^{1/4}\theta_0'(0), \tag{4.13}$$

using equations (3.8), (4.5) and (4.8), where λ is the thermal conductivity. The numerical integration gave $\theta_0'(0) = 0{\cdot}0036$.

If $(T-T_\infty)/T_\infty \gg$ (Mach number)^2 at all points in the wall jet, then the effects of viscous heating may be neglected and the particular integral (4.5) may be taken as $\bar{T} = 1$.

Wall heating

To describe the effects on the temperature distribution of maintaining the wall at a constant temperature $T_w \neq T_\infty$, we require a complementary function (4.6) with $\alpha = 0$. Thus

$$\bar{T}_1 = (\bar{T}_w - 1)\theta_1(\eta), \tag{4.14}$$

where

$$\left.\begin{aligned}\theta_1(0) &= 1,\\ \theta_1(\infty) &= 0.\end{aligned}\right\} \tag{4.15}$$

The required solution of equation (4.7) is immediately obtained as

$$\theta_1(\eta) = \int_\eta^\infty \exp\left\{-\sigma\int_0^\eta f\,d\eta\right\}d\eta\bigg/\int_0^\infty \exp\left\{-\sigma\int_0^\eta f\,d\eta\right\}d\eta. \tag{4.16}$$

As shown by Glauert (1956), $f = g^2$, where $g' = \frac{1}{3}(1-g^3)$, and hence

$$-\int_0^\eta f\,d\eta = \log(1-g^3), \tag{4.17}$$

and equation (4.16) may be written

$$\theta_1 = \int_g^1 (1-g^3)^{\sigma-1}\,dg\bigg/\int_0^1 (1-g^3)^{\sigma-1}\,dg. \tag{4.18}$$

When $\sigma = 1$ this reduces to

$$\theta_1 = (1-g). \tag{4.19}$$

When $\sigma \neq 1$, θ_1 may be represented as an incomplete beta function by writing $(1-g^3) = t$ in equation (4.18), which then becomes

$$\theta_1 = \int_0^t t^{\sigma-1}(1-t)^{-2/3}\,dt\bigg/\int_0^1 t^{\sigma-1}(1-t)^{-2/3}\,dt$$

$$= B_t(\sigma, \tfrac{1}{3})/B(\sigma, \tfrac{1}{3}). \tag{4.20}$$

The contribution to the rate at which heat is transferred across the wall, per unit area, is given by

$$q_w = -\tfrac{3}{4}\lambda_\infty(T_w - T_\infty)\left(\frac{\nu_\infty x^4}{U(x^3+l^3)^3}\right)^{1/4}\theta_1'(0), \tag{4.21}$$

from equations (3.8), (4.12) and (4.14). Expressing the beta function in terms of Γ functions, we find that $\theta_1'(0) = -0{\cdot}2861$.

Initial heating

When the jet is heated initially, we require another complementary function of the form (4.6),

$$T_2 = C_2(\bar{x}^3 + l^3)^{-\alpha}\theta_2(\eta) \tag{4.22}$$

where $\theta_2(\eta)$ satisfies equation (4.7). The boundary conditions are

$$\theta_2(\infty) = 0 \tag{4.23}$$

and either $\theta_2(0) = 0$ for constant wall temperature, (4.24)

or $\theta_2'(0) = 0$ for a thermally insulated wall. (4.25)

The determination of α is an eigenvalue problem. We shall consider first the case of a thermally insulated wall. Integrating equation (4.7) with respect to η between the limits 0 and η, and using (3.3) and (4.25) we have

$$\theta_2' + \sigma\left\{\theta_2 f + (4\alpha - 1)\int_0^\eta f'\theta_2\,d\eta\right\} = 0. \tag{4.26}$$

On letting $\eta \to \infty$, equation (4.26) becomes, from (4.23),

$$(4\alpha - 1)\int_0^\infty f'\theta_2\,d\eta = 0. \tag{4.27}$$

Unless the integral in (4.27) is zero, which on physical grounds is unlikely, this shows that $\alpha = \frac{1}{4}$. With this value of α, equation (4.7) may be integrated in closed terms, using the result (4.17), to give

$$\theta_2 = (1 - g^3)^\sigma. \tag{4.28}$$

It follows that the contribution to the wall temperature is

$$T_{2w} = C_2(\bar{x}^3 + l^3)^{-1/4}. \tag{4.29}$$

The value of the constant C_2, which depends on the initial conditions in the jet, is discussed later.

For a wall maintained at constant temperature, it may be verified that the solution of (4.7) satisfying the boundary conditions (4.23) and (4.24) is given by

$$\alpha = \frac{(3\sigma + 1)}{8\sigma}, \tag{4.30}$$

$$\theta_2 = g(1 - g^3)^\sigma. \tag{4.31}$$

When $\sigma = 1$, these equations become $\alpha = \frac{1}{2}$, $\theta_2 = \frac{3}{2}f'$, in accord with the term Bu in equation (4.1). The contribution to the rate at which heat is transferred across the wall, per unit area, is given by

$$q_w = -\tfrac{3}{4}\lambda_\infty T_\infty C_2\left[\left(\frac{\nu_\infty}{U}\right)^{(11\sigma+3)/2\sigma} \frac{x^4}{(x^3 + l^3)^{(9\sigma+1)/2\sigma}}\right]^{1/4} \theta_2'(0), \tag{4.32}$$

from equations (3.8), (4.12) and (4.22), where, from (4.31), $\theta_2'(0) = \frac{1}{3}$.

An estimate of the constant C_2 may be obtained as follows. The maximum values of θ_2 are 1 in equation (4.28), and 0·4740 in equation (4.31), when $\sigma = 0{\cdot}72$. Hence, by requiring the maximum temperature at $\bar{x} = 0$

(or $\bar{x} = \bar{a}$ when the jet flows from beneath a circular disc) to be the excess of total temperature at this point in the physical flow, the value of C_2 may be roughly determined.

Since the equation (4.4) is linear, we may construct a solution which is any linear combination of the solutions obtained above, and thus we are able to deal with a wall jet of arbitrary Mach number, wall temperature and initial heating. When the wall is maintained at constant temperature we require (4.5), (4.14) and (4.22), the total rate at which heat is transferred across the wall, per unit area, being given by a combination of (4.13), (4.21) and (4.32). For a thermally insulated wall (4.5) and (4.22) are the appropriate solutions, the wall temperature being given by a combination of (4.11) and (4.29).

5. Results

The results obtained in § 3 and § 4 are presented in figures 1–5. Figure 1 shows the velocity function $f'(\eta)$ whilst figures 2–5 show the temperature distribution across the jet and the resulting dependence of the geometrical distance y on η for all the cases considered above. This dependence of y on η is given directly by equation (3.8) when the temperature distribution is known. Since the velocity u is a function of the mass flux ψ, a decrease

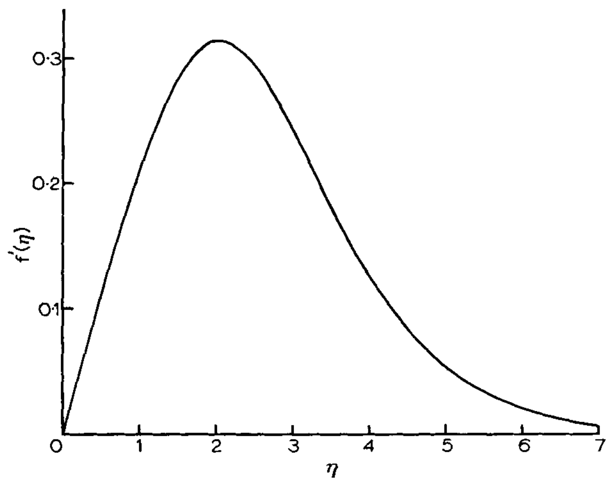

Figure 1. The velocity function $f'(\eta)$.

of ρ (corresponding to an increase of T) causes a physical broadening of the wall jet, and an increase of ρ causes a narrowing. The broadening or narrowing occurs most where $|T - T_\infty|$ is greatest. Since, in general, the temperature distribution across the wall jet changes as the jet develops, it follows that even for our similarity solution the geometrical form of the velocity profile is not independent of x, as it is in the corresponding incompressible flow.

The effect of viscous heating is shown in figure 2 where the temperature distributions across the wall jet, for a thermally insulated wall and for a wall maintained at constant temperature T_∞, are given from equation (4.5). The simple Crocco relation for $\sigma = 1$, given by equation (4.1) with $B = 0$, is included for comparison. In this case the total energy in the wall jet is constant, so the effect of a decrease in velocity is exactly balanced by an

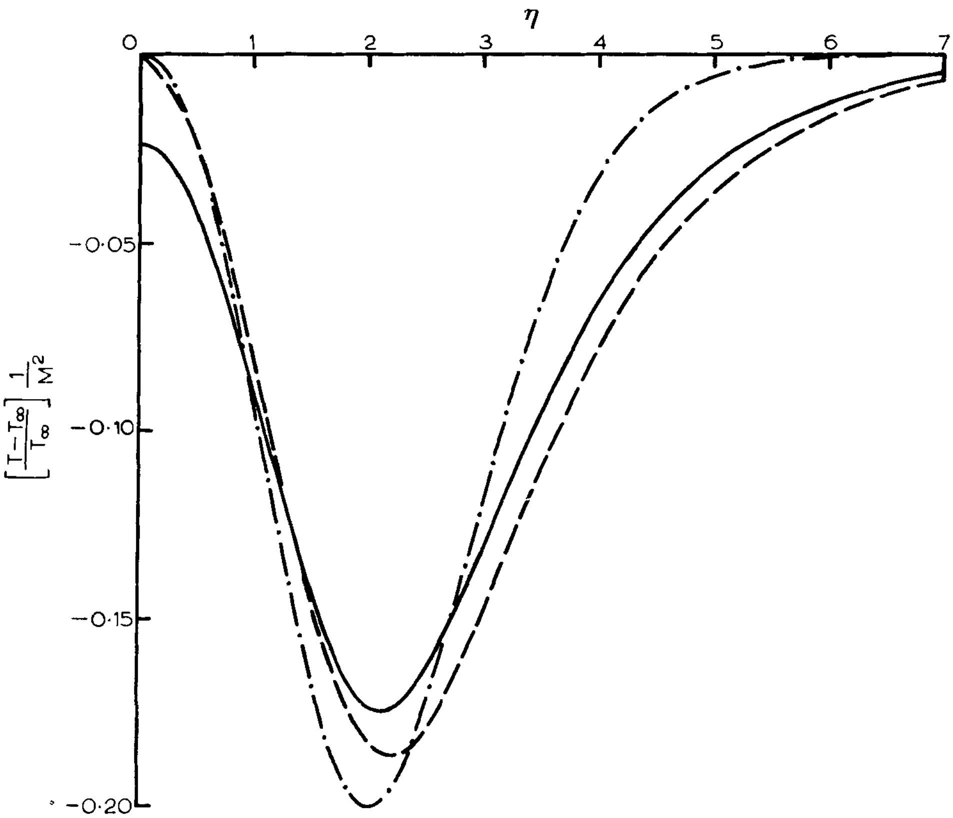

Figure 2. The temperature distribution due to viscous heating;
———— thermally insulated wall,
— — — — wall maintained at constant temperature T_∞,
—·—·— Crocco solution for $\sigma = 1$.

increase in temperature. When the wall is maintained at a temperature T_∞ there is no heat transfer across the wall and hence both conditions at the wall are satisfied together. For $\sigma = 0{\cdot}72$ this energy balance is not maintained, and a solution for each of the wall conditions is required; qualitatively, however, the results are similar to the case $\sigma = 1$. Figure 3 shows how the wall jet is narrowed in the region of higher velocity, in comparison with the profile at low Mach number. The Mach number M is here defined as the ratio of the maximum velocity across the wall jet to the velocity of sound outside the wall jet.

When the wall is maintained at a constant temperature $T_w \neq T_\infty$, the contributions to the temperature distribution and to the variation of y with η are as shown in figure 4. These results were obtained by numerically integrating equation (4.20) for the value $\sigma = 0{\cdot}72$. As we might expect, the temperature increment is greatest near the wall, gradually falling to zero with increasing distance from the wall.

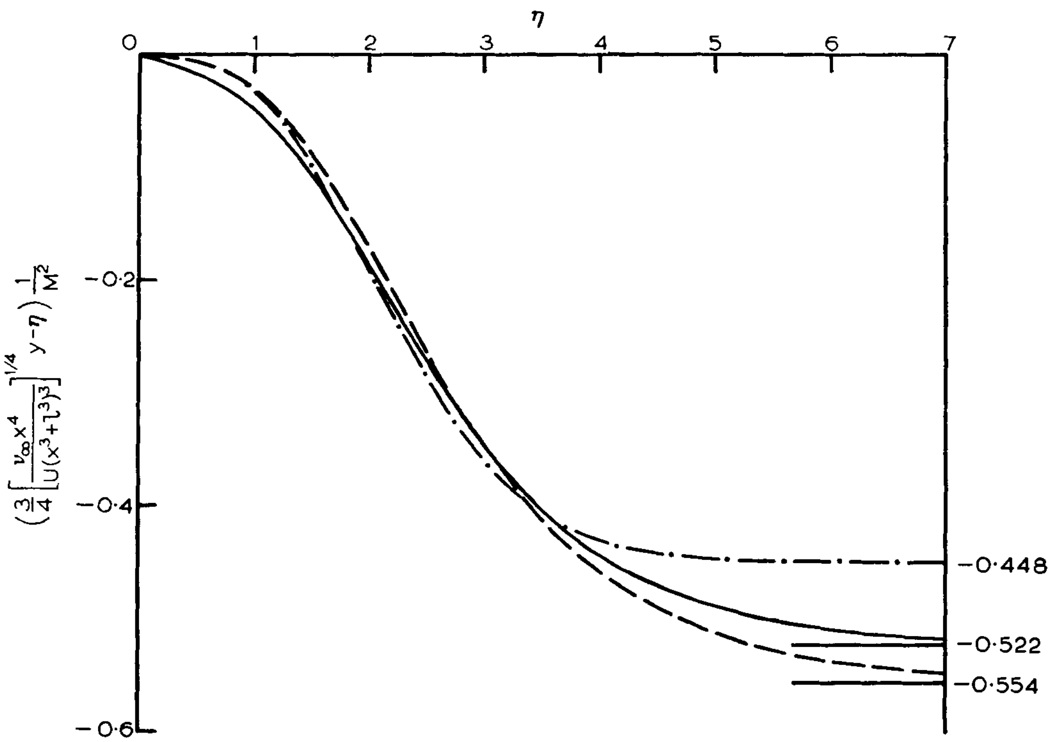

Figure 3. The variation of y with η due to viscous heating;
———— thermally insulated wall,
— — — — wall maintained at constant temperature T_∞,
—·—·—·— Crocco solution for $\sigma = 1$.

Figure 5 depicts the contribution to the temperature distribution and to the variation of y with η when heat is added initially to the jet, for the two cases of a thermally insulated wall and a perfectly conducting wall. For a thermally insulated wall the effects of the added heat are similar to those shown in figure 4, though the variation with distance from the axis is different. For a perfectly conducting wall, the effects of the added heat are mainly confined to the faster moving regions of the wall jet.

As we have seen in § 4, when a hot wall jet flows at high speeds over a heated wall, the temperature distribution can be obtained by taking a suitable linear combination of the distributions that have been considered above. Similarly, we see from equation (3.8) that to obtain the total variation of y with η we need the same combination of the appropriate distributions shown

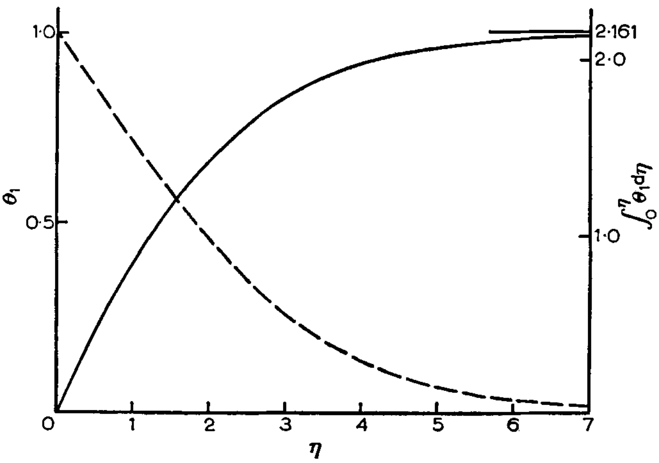

Figure 4. Contributions to the temperature distribution (θ_1 — — — —), and to the variation of y with $\eta\left(\int_0^\eta \theta_1\,d\eta\right.$ ————$\left.\right)$, when the wall is maintained at constant temperature $T_w \neq T_\infty$.

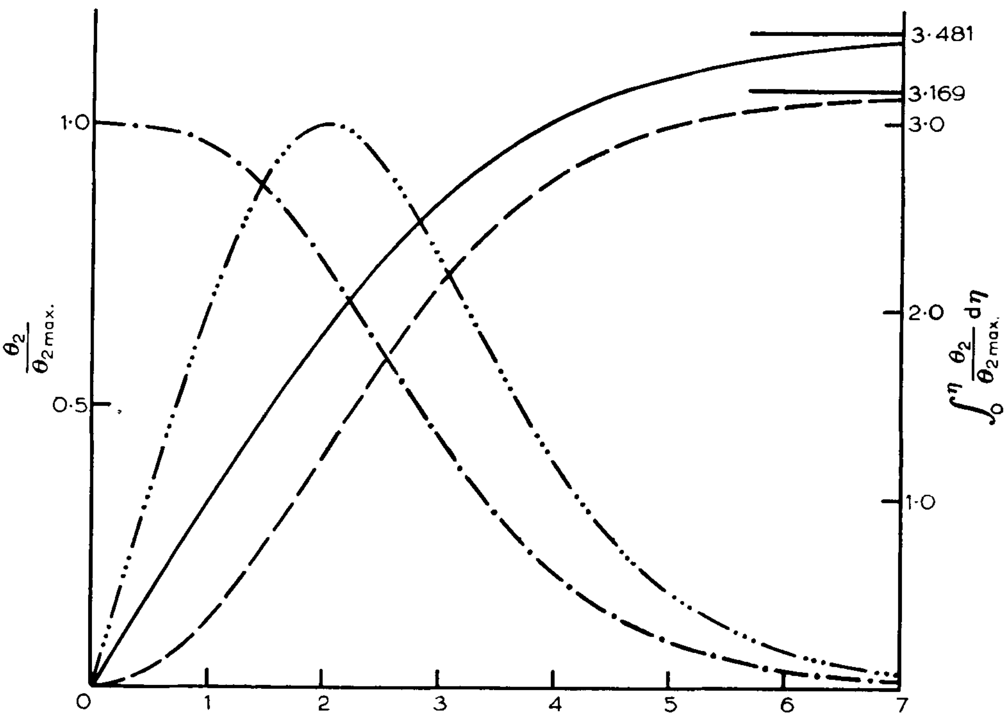

Figure 5. Contributions to the temperature distribution ($\theta_2/\theta_{2\max}$) and to the variation of y with η $\left(\int_0^\eta \theta_2/\theta_{2\max}\,d\eta\right)$, when heat is added to the jet initially; — · — · — · —, ———— for a thermally insulated wall respectively, — · · · — · · · —, — — — — and for constant wall temperature respectively.

in figures 3–5, the total increment in y being the sum of the increments of y in the separate distributions. As the distance from the axis of the jet increases, the Mach number decreases and the added heat is diffused away until their effects on the temperature eventually become negligible. We see from equation (4.5) that the effects of viscous heating, which fall off like $(x^3+l^3)^{-1}$, decrease faster than the effects of added heat which, from (4.22), behave like $(x^3+l^3)^{-\alpha}$, where $\alpha = 0{\cdot}25$ for a thermally insulated wall, and $\alpha = 0{\cdot}548$ for a perfectly conducting wall when $\sigma = 0{\cdot}72$. However, the effect of a wall temperature $T_w \neq T_\infty$ does not decay with increasing x. As a result the form of the velocity profile changes as the wall jet develops, and only when the effects of viscous heating and added heat have become negligible is geometrical similarity achieved, the final velocity profile being the incompressible one unless the wall is maintained at a temperature different from T_∞.

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